

A century after Griffith: towards a combined non-linear criterion for brittle failure

Technical Appendix

Complete mathematical derivations,
computational algorithms and R implementation

Supplementary material accompanying:

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Introduction

This document provides the mathematical developments and computational resources that support the manuscript “A century after Griffith: towards a combined non-linear criterion for brittle failure”. In order to maintain the readability of the main paper, several analytical derivations, intermediate developments and implementation details have been omitted from the published manuscript and are presented here in full.

The supplementary material is organised into three sections. Section A revisits the original Griffith criterion and develops its formulation in Mohr space. Section B presents the complete analytical derivation of the New Griffith Criterion (NGC), including alternative formulations and particular cases. Section C contains the R scripts used to reproduce the calculations and figures presented in the manuscript.

The most recent version of the R code, together with an interactive application for generating the NGC and Ücar failure envelopes, is maintained in the accompanying [GitHub repository](#).

A Original Griffith Criterion

The derivation presented in this section closely follows Griffith’s original formulation, while adopting a notation consistent with that used throughout the main manuscript.

The Griffith’s criterion (OGC) for crack propagation is based on the following principle:

A pre-existing, favourably oriented flaw will propagate when the reduction in elastic strain energy equals or exceeds the energy required to create the new surface area of the crack. In his model, he considered an elliptical flaw or imperfection with a major semi-axis of dimension a and with eccentricity close to unity. Under these conditions, he determined—using Eq.A.1—the energy stored due to the presence of a crack of width t subjected to a uniform remote tensile stress σ .

$$U_1 = \frac{\pi \cdot \sigma^2 \cdot a^2 \cdot t}{E} \quad (\text{A.1})$$

where E is the modulus of elasticity of the material under consideration. On the other hand, the stored mechanical strain energy of a plate, in absence of fissures, of volume V (length l , width w and thickness t) is:

$$U_2 = \frac{\sigma^2 \cdot V}{2 \cdot E} \quad (\text{A.2})$$

In these conditions, the strain energy release is:

$$U = (U_2 - U_1) = \frac{\sigma^2 \cdot V}{2 \cdot E} - \frac{\pi \cdot \sigma^2 \cdot a^2 \cdot t}{E} \quad (\text{A.3})$$

The energy input required to break atomic bonding and allow two new surfaces:

$$U_s = 2 \cdot (2 \cdot a \cdot t \cdot \gamma_s) \quad (\text{A.4})$$

In the latter equation, U_s is the surface energy and represents a measure of the resistance for crack extension. Additionally, γ_s is the specific fracture energy, whose values in various rocks are typically on the order of 10 J/m^2 (i.e., 10^4 erg/cm^2), and is a material property.

Considering Eqs.A.3 and A.4, the total energy in the presence of the crack and under external loading is:

$$U_t = (U_2 - U_1 + U_s) = \left(\frac{\sigma^2 V}{2E} - \frac{\pi \sigma^2 a^2 t}{E} \right) + 4at\gamma_s \quad (\text{A.5})$$

Under these conditions, brittle fracture will occur as a result of the competition between the strain energy released and the surface energy required to create new fracture surfaces. Therefore, crack propagation will take place when the rate of strain energy release equals the rate of surface energy expenditure. This condition can be expressed as follows:

$$\frac{d(4a\gamma_s)}{da} - \frac{d}{da} \left(\frac{\pi \sigma^2 a^2}{E} \right) = 0 \quad (\text{A.6})$$

Based on the above, by differentiating Eq.A.6 with respect to the major semi-axis a , and assuming a plane stress condition, the failure stress for crack propagation is obtained as:

$$\sigma_f = \sqrt{\frac{2E\gamma_s}{\pi a}} \quad (\text{A.7})$$

For values of stress lower than σ_f , the crack will not propagate, since the energy released by a potential advance of the flaw will be insufficient to generate new fracture surfaces.

From Eq.A.6, it follows that the criterion for crack propagation—i.e., the energy release rate of the system—must satisfy:

$$\frac{dU_1}{da} \geq 4\gamma_s \quad (\text{A.8})$$

The application of the above for the case of rocks, leads to the OGC:

$$\left. \begin{aligned} \sigma_3 &= T_0 & , \text{ if: } \sigma_1 &\leq -3 \cdot \sigma_3 \\ (\sigma_1 - \sigma_3)^2 &= -8 \cdot T_0 \cdot (\sigma_1 + \sigma_3) & , \text{ if: } \sigma_1 &\geq -3 \cdot \sigma_3 \end{aligned} \right\} \quad (\text{A.9})$$

That is, in the pure tensile domain, the failure envelope reduces to a vertical line at $\sigma_3 = T_0$, whereas in the transition regime it takes the form of a conic section, specifically a parabola. From the second part of Eq.A.9 it follows that:

$$\frac{\sigma_1}{|T_0|} = \left(\frac{\sigma_3}{|T_0|} + 4 \right) + 4\sqrt{1 + \frac{\sigma_3}{|T_0|}}, \text{ for } T_0 \leq \sigma_3 \leq 0 \quad (\text{A.10})$$

Therefore, from Eq. A.10 when $\sigma_3 = T_0$, then $\sigma_1 = 3|T_0|$, and, as shown in Fig. A.1 (fig.1 of the main text), point $P(\sigma_3, \sigma_1) = P_1(T_0, 3|T_0|)$ defines the starting point of the failure parabola.

A.1 OGC in the Mohr space

The OGC can be reformulated in the (σ, τ) space, instead of being expressed in terms of the principal stresses (σ_3, σ_1) as typically required for Mohr diagram analysis. This reformulation is achieved through a change of variables; the equation of the failure envelope, which is defined by a system of equations, is obtained in explicit form as indicated by Jaeger and Cook (1969), yielding the following expressions:

$$\frac{\tau}{|T_0|} = 2\sqrt{\left(\frac{\sigma}{|T_0|} + 1\right)}, \text{ for } T_0 \leq \sigma \leq 2|T_0| \quad (\text{A.11})$$

It is important to note that eq. A.11 also defines a parabola that is symmetric with respect to the normal stress axis:

$$\sigma = \sigma_3 + 2|T_0|\sqrt{1 + \frac{\sigma_3}{|T_0|}} \quad (\text{A.12})$$

On the other hand, the value of $\sigma = f(\sigma_3)$ is obtained through analytical procedures that allow the failure envelope of a family of curves to be determined using the same procedure described in Section 5.2. Under these conditions, by combining Eq. A.10 with Eq. B.26 from the aforementioned section and applying the appropriate algebraic manipulations, we obtain:

$$\sigma = \sigma_3 + 2|T_0|\sqrt{1 + \frac{\sigma_3}{|T_0|}} \quad (\text{A.13})$$

For the particular case where $\sigma_3 = 0$, the corresponding values of the normal stress and shear strength simplify to:

$$\begin{aligned} (\tau)_{\sigma_3=0} &= \sqrt{3}\sigma = 2|T_0|\sqrt{3} \\ (\sigma)_{\sigma_3=0} &= 2|T_0| \end{aligned} \quad (\text{A.14})$$

Alternatively, the quadratic expression for the failure envelope in Equation A.11 can be written in the following equivalent form:

$$\tau = 2 \cdot |T_0|\sqrt{\frac{\sigma}{|T_0|} + 1} \quad (\text{A.15})$$

It follows directly from Equation A.15 that the shear strength at the origin—that is, for zero normal stress—is $2 \cdot |T_0|$. Furthermore, the slope of the parabola at this point can be determined analytically as:

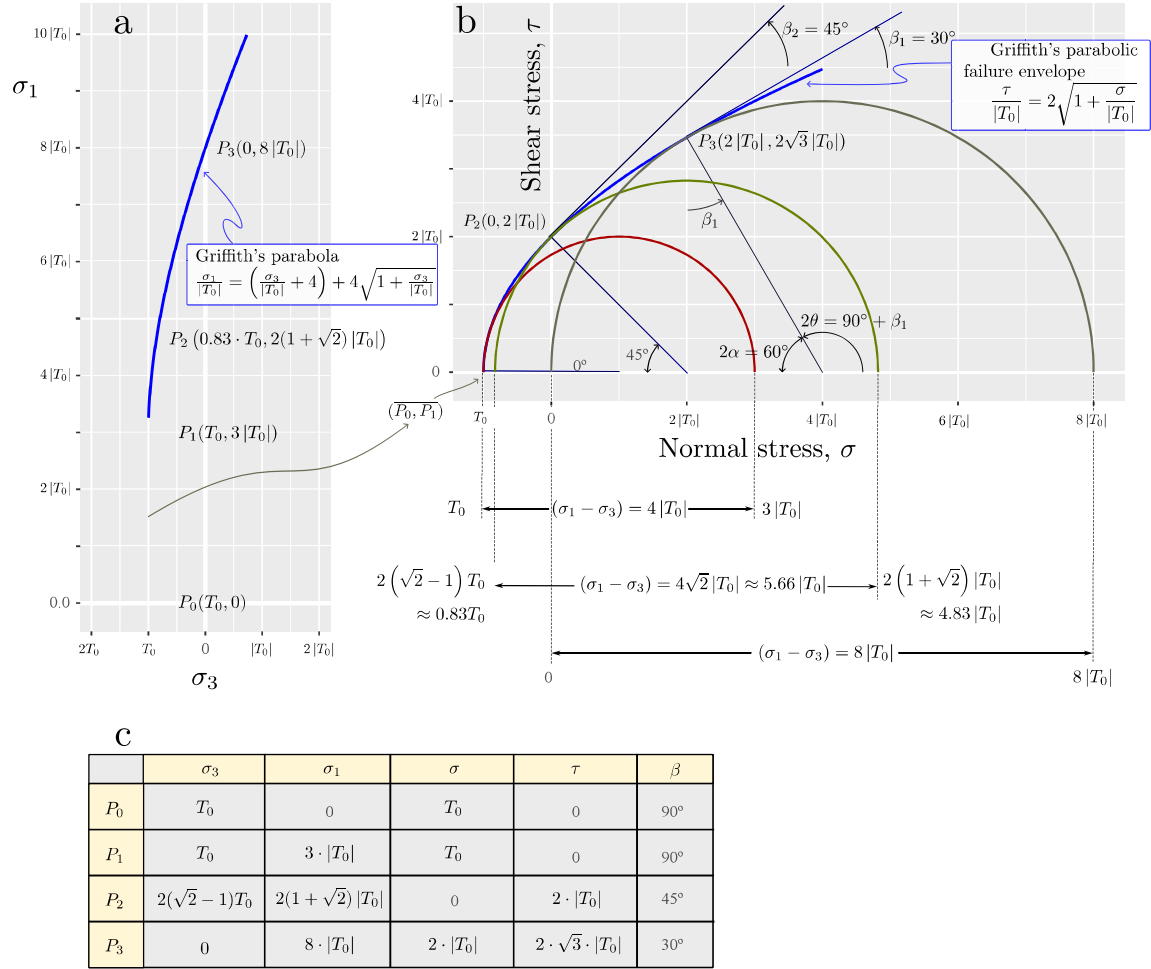


Figure A.1: Graphical expression of OGC in terms of stresses. A. OGC failure parabola in the $(\sigma_1 - \sigma_3)$ space, normalized expressed by scaling with the tensile strength, T_0 . Pairs (σ_1, σ_3) that fall on the segment $(\bar{P}_0, \bar{P}_1)$ will produce pure tensional fractures, orthogonal to σ_3 . Pairs (σ_1, σ_3) touching the curve between P_1 and P_2 will produce oblique extension fractures formed under a negative normal stress σ . The curve segment P_2 to P_3 marks the formation of fractures under positive normal stress, albeit the minimum main stress, σ_3 , is still negative. From P_3 onwards, is the dominion of shear fractures on compression, under positive main stresses. B. OGC failure parabolic envelope in the (τ, σ) space. The pairs (σ_1, σ_3) are now circles, that when in contact with the failure envelope indicate fracturing of the rock. Three of such circles are represented in detail, with their respective values of σ , τ and the diameter of the circle $(\sigma_1 - \sigma_3)$. From left to right, the first circle represents the stress state with maximum σ_1 that still yields pure tensional fractures. The second one corresponds to the stress state for which $\sigma = 0$. The third circle corresponds to the uniaxial compressive test ($\sigma_3 = 0$, $\sigma_1 = C_0$), and is the smallest stress differential that generates shear fractures fully in compression.

$$\tan \beta = \left(\frac{d\tau}{d\sigma} \right) = \frac{1}{\sqrt{\frac{\sigma}{|T_0|} + 1}} \quad (\text{A.16})$$

Using Equation A.16, together with Figure A.1 (fig.2 of the main text), it is possible to evaluate the slope of the failure envelope at the point where the curve intersects the vertical axis, namely at $P(0, 2|T_0|)$. This yields:

$$\tan \beta = \left(\frac{d\tau}{d\sigma} \right)_{\sigma=0} = 1 \Rightarrow \beta = 45^\circ \quad (\text{A.17})$$

The OGC yields satisfactory results under tensile stress conditions. However, under a compressive regime, the curvature of the parabola flattens, leading to an underestimation of the actual strength at failure, as evidenced by triaxial test results.

It is well known that one of the characteristics of empirical failure criteria in rocks is that the failure envelope may yield satisfactory results under shear in the compressive domain, that is, for values of $\sigma_3 \geq 0$. However, in contrast with the OGC, a significant limitation of these criteria is their inability to provide reliable results in the tensile region. This region initially consists of pure tension (segment $(\overline{P_0}, \overline{P_1})$ in fig.A.1, where the minimum principal stress $\sigma_3 = T_0$ is equal to the normal stress σ acting perpendicularly to the failure plane. Additionally, it includes a transitional hybrid zone characterised by a stress field combining tension and shear, for values of $\sigma_3 \geq (2\sqrt{2} - 2) \cdot T_0$, that is to say $\sigma_3 = -0.83 \cdot |T_0|$. From fig. A.1 let us remember that when $\sigma_3 = 0.83 \cdot T_0$ then $\sigma = 0$. Therefore, the region between P_2 and P_3 corresponds to values such that $0.83 \cdot T_0 \leq \sigma_3 \leq 0$, that is to say, $\sigma \geq 0$.

In this context, when employing the linear Mohr-Coulomb criterion, the tensile region is represented by a vertical line defined as $\sigma_3 = T_0$ (tension cut-off), which intersects the linear Mohr-Coulomb envelope. Under these conditions, it is possible to approximate the stress values in the tensile region. This approximation is necessary because the Mohr-Coulomb criterion does not account for frictional resistance under tensile stress conditions and tends to overestimate the tensile strength.

Furthermore, it is common practice to assign the value $\tau = 2|T_0|$ as the shear strength at zero normal stress. This magnitude is directly related to the Griffith failure criterion (Griffith, 1924), along with the associated ratio $(C_0/|T_0|) = 8$. In this context, it is worth highlighting that when the Úcar criterion (Úcar, 2021) is applied, the resulting shear strength values are very close to $2|T_0|$ when the ratio $C_0/|T_0|$ is approximately 10.

On the other hand, as previously indicated, and taking as an example the case where $(C_0/|T_0|) = 22$, the resulting value is $\tau \approx 3|T_0|$. This observation indicates that the shear or cohesive strength in the absence of normal stress increases as the ratio $C_0/|T_0|$ becomes larger.

In the case of the Hoek–Brown failure criterion, it is recommended that in the tensile region, the generalised Griffith criterion proposed by Fairhurst (1964) be used. This criterion is expressed as a function of the ratio $(|T_0|/C_0)$.

B New Griffith Criterion

Taking all of the previous considerations and factors into account, an expression has been developed that depends on the parameter $\xi = (T_0/C_0)$, incorporating certain boundary conditions in order to determine the constants involved in the proposed equation. Also, note that the tensile strength T_0 is negative.

B.1 Analytical Development

This section describes and justifies the modifications to the OGC, transforming Eq.A.9, as stated in the main text, to

$$\left. \begin{aligned} \sigma_3 &= T_0 && , \text{ a vertical line, when: } \sigma_1 \leq -\lambda \cdot T_0 \\ \sigma_1 &= a_1 \cdot (\sigma_3 - T_0)^{3/4} + \lambda \cdot |T_0| && , \text{ a plane curve, when: } \sigma_1 \geq -\lambda \cdot T_0 \end{aligned} \right\} \quad (\text{B.1})$$

here, we outline the procedure used to determine the constants involved in the analytical expressions that define the relationship between the principal stresses in the pure tensile region and the hybrid (tensile-to-shear transition) region.

By expressing the plane curve of the second part of Eq.B.1 in dimensionless form with respect to C_0 , it becomes:

$$\left(\frac{\sigma_1}{C_0} \right) = \frac{a_1}{C_0^{1/4}} \left(\frac{\sigma_3}{C_0} - \frac{T_0}{C_0} \right)^{3/4} + \lambda \frac{|T_0|}{C_0} \quad (\text{B.2})$$

if we call $a = a_1/(C_0^{1/4})$, $\bar{\sigma}_1 = \frac{\sigma_1}{C_0}$ and $\bar{\sigma}_3 = \frac{\sigma_3}{C_0}$ it results in the following expression for the NGC:

$$\bar{\sigma}_1 = a (\bar{\sigma}_3 - \xi)^{3/4} + \lambda |\xi| \quad (\text{B.3})$$

By considering the condition under which $\sigma_3 = 0$, it follows that $(\sigma_1/C_0) = 1$. Therefore, eq.B.3) simplifies to the following expression:

$$a(-\xi)^{3/4} + \lambda |\xi| = 1 \quad (\text{B.4})$$

On the other hand, the slope of the failure envelope in the plane is given by:

$$\left(\frac{d\bar{\sigma}_1}{d\bar{\sigma}_3} \right) = \tan^2 \theta = \frac{3a}{4} \cdot \frac{1}{(\bar{\sigma}_3 - \xi)^{1/4}} \quad (\text{B.5})$$

$$\tan \theta = \sqrt{\frac{3a}{4} \cdot \frac{1}{(\bar{\sigma}_3 - \xi)^{1/4}}} = \tan \left(45^\circ + \frac{\beta}{2} \right) \quad (\text{B.6})$$

Applying again the condition $\sigma_3 = 0$, the above expression can be rewritten as follows:

$$\tan \theta = \tan \theta_1 = \sqrt{\frac{3a}{4} \cdot \frac{1}{(-\xi)^{1/4}}} \quad (\text{B.7})$$

Furthermore, using the notation $(\theta)_{\sigma_3=0} = \theta_1 = \left(45^\circ + \frac{\beta_1}{2}\right)$, the relationship between the parameter a and the angle β_1 , which defines the tangent to the failure envelope at the point $P_1(\sigma, \tau)$ when $\sigma_3 = 0$, is therefore:

$$\overbrace{\sqrt{\frac{3a}{4} \frac{1}{(-\xi)^{1/4}}}}^{\tan \theta_1} - \tan \left(45^\circ + \frac{\beta_1}{2}\right) = 0 \quad (\text{B.8})$$

It is important to note that the correct determination of the parameters a and λ must satisfy three boundary conditions:

The first condition is derived from the failure envelope, where the shear strength is $\tau = 0$ when the minimum principal stress is $\sigma_3 = \sigma = T_0$, and the angle is $\beta = 90^\circ$. Under these conditions, the dimensionless normal stress is given by $\bar{\sigma} = \xi = \frac{T_0}{C_0}$. This condition is detailed in section B.2 through eqs. B.56 and B.58, which represent the normal stress and shear strength, both in parametric form as functions of the angle β .

The second condition corresponds to the case where $\sigma_3 = 0$, and is represented by equ. B.4.

Since only two equations are available for three unknowns—namely a , λ , and β_1 —it is necessary to introduce a third boundary condition to ensure the system of nonlinear equations is well-posed.

The third condition to be satisfied occurs when $\sigma = 0$.

Therefore, the constants a and λ must be intrinsically related to this condition and to the other parameters involved.

Continuing with the analytical procedure, and applying the equation that represents the normal stress acting on the failure plane, the following expression is obtained:

$$\left(\frac{\sigma}{C_0}\right) = \bar{\sigma}_n = \bar{\sigma}_1 \cos^2 \theta + \bar{\sigma}_3 \sin^2 \theta \quad (\text{B.9})$$

Based on the aforementioned considerations, and assuming $\sigma = 0$, eq. B.9 becomes:

$$\tan^2 \theta = -\frac{\bar{\sigma}_1}{\bar{\sigma}_3} \quad (\text{B.10})$$

Moreover, we also have:

$$\bar{\sigma}_3 = (\bar{\sigma}_3)_{\sigma=0} \quad \text{and} \quad (\theta)_{\sigma=0} = \theta_2 = \left(45^\circ + \frac{\beta_2}{2}\right)$$

Additionally, β_2 is the angle formed by the tangent to the failure envelope at the point where $\sigma = 0$. Under these conditions, the third equation is directly related to the parameters a and λ , and is obtained by substituting eqs.B.3 and B.6 into eq.B.10:

$$\frac{3a}{4} \frac{1}{(\bar{\sigma}_3 - \xi)^{1/4}} + \frac{a(\bar{\sigma}_3 - \xi)^{3/4} + \lambda|\xi|}{\bar{\sigma}_3} = 0 \quad (\text{B.11})$$

Which is equivalent to writing:

$$3a\bar{\sigma}_3 + 4a(\bar{\sigma}_3 - \xi) + 4(\bar{\sigma}_3 - \xi)^{1/4} \cdot \lambda|\xi| = 0 \quad (\text{B.12})$$

On the other hand, considering that:

$$\bar{\sigma}_3 = (\bar{\sigma}_3)_{\sigma=0} \quad \text{and} \quad (\beta)_{\sigma=0} = \beta_2$$

Eq.B.6, in terms of the aforementioned parameters, takes the following form:

$$\sqrt{\frac{3a}{4} \frac{1}{(\bar{\sigma}_3 - \xi)^{1/4}}} - \tan \left(45^\circ + \frac{\beta_2}{2} \right) = 0 \quad (\text{B.13})$$

This represents **the fourth condition** for the equation.

Based on the previous discussion, we now have four equations, namely B.4, B.8, B.12, and B.13, and five unknowns: a , λ , $(\beta_1)_{\sigma_3=0}$, $(\bar{\sigma}_3)_{\sigma=0}$, and $(\beta_2)_{\sigma=0}$.

It is worth noting that the value of $(\bar{\sigma}_3)_{\sigma=0}$ is particularly relevant because of its proximity to T_0 , typically approaching a value close to 7/8 of T_0 .

The fifth equation required to complete the system corresponds to the normal stress derived from the differential equation in conjunction with the failure envelope fitted to the Mohr circles. Additionally, as will be shown later, both stress components are expressed parametrically as a function of the angle β . For further details, refer to Appendix B.2 (eqs.B.56 and B.60).

Thus, the normal stress is given by:

$$\bar{\sigma}_n = \xi + (1 - \sin \beta) [\Omega_1 - 7\Omega I] \quad (\text{B.14})$$

Considering that:

$$\bar{\sigma}_n = 0, \quad (\beta)\sigma = 0 = \beta_2 \quad (\text{B.15})$$

The previous equation becomes:

$$\xi + (1 - \sin \beta_2) [\Omega_1 - 7\Omega I] = 0 \quad (\text{B.16})$$

Where:

$$\Omega = \left(\frac{3a}{4}\right)^4 \quad (\text{B.17})$$

Additionally, I is a parameter that depends on the angle β , and Ω_1 corresponds to the integration constant, which can be determined by considering the condition when $\sigma_3 = 0$ and $\beta = \beta_1$, as described in detail in Appendix B.2.

Considering the latter condition, we obtain:

$$(\Omega_1)_{\beta=\beta_1} = \frac{1}{2} + 7I \left(\frac{3a}{4}\right)^4 - \frac{\xi}{(1 - \sin \beta_1)} \quad , \quad I = f(\beta_1) \quad (\text{B.18})$$

Therefore, by relating β_1 and β_2 , eq.B.16 can be written as:

$$\xi + (1 - \sin \beta_2) \left[(\Omega_1)_{\beta=\beta_1} - 7 \left(\frac{3a}{4}\right)^4 (I)_{\beta=\beta_2} \right] = 0 \quad (\text{B.19})$$

In conclusion, by incorporating this last equation, the system consists of five unknowns: a , λ , β_1 , β_2 , and $(\bar{\sigma}_3)_{\sigma=0}$, and five equations: B.4, B.8, B.12, B.13, and B.19.

To summarize the analysis, the annexed table provides a comprehensive overview of the analytical procedure followed.

Boundary condition	Resulting equation
$\bar{\sigma}_3 = 0, \sigma_1 = C_0$	$1 - a(-\xi)^{3/4} - \lambda \xi = 0$
$\bar{\sigma}_3 = 0, \beta = \beta_1$	$\sqrt{\frac{3a}{4} \frac{1}{(-\xi)^{1/4}}} - \tan\left(45^\circ + \frac{\beta_1}{2}\right) = 0$
$\bar{\sigma}_3 = (\bar{\sigma}_3)_{\sigma=0}, \beta = \beta_2$	$3a\bar{\sigma}_3 + 4a(\bar{\sigma}_3 - \xi) + 4(\bar{\sigma}_3 - \xi)^{1/4} \cdot \lambda \xi = 0$
$\bar{\sigma}_3 = (\bar{\sigma}_3)_{\sigma=0}, \beta = \beta_2$	$\sqrt{\frac{3a}{4} \frac{1}{(\bar{\sigma}_3 - \xi)^{1/4}}} - \tan\left(45^\circ + \frac{\beta_2}{2}\right) = 0$
$I = f(\beta)$	$\xi + (1 - \sin \beta_2) \left[(\Omega_1)_{\beta=\beta_1} - 7 \left(\frac{3a}{4}\right)^4 (I)_{\beta=\beta_2} \right] = 0$

Table B.1: Resume of the boundary conditions and the resulting equations to determine the needed parameters: a , λ , β_1 (β when $\sigma_3 = 0$), β_2 (β when $\sigma = 0$), and $(\bar{\sigma}_3)_{\sigma=0}$.

The equations presented in TableB.1 must be solved subject to the following constraints:

$$\left. \begin{aligned} &\xi \leq \bar{\sigma}_3 < 0, \sigma_3 = \frac{7}{8}C_0 : \text{recommended seed value for iterative search} \\ &2 \leq a \leq 10, a \approx 0.531 \cdot \left(\frac{C_0}{|T_0|}\right)^{3/4} : \text{recommended seed value for iterative search} \\ &3 \leq \lambda \leq 15, \lambda = 0.472 \cdot \frac{C_0}{|T_0|} \\ &0.45 \leq \beta_1 \leq 0.9 \text{ (angles in radians)} \\ &\beta_1 < \beta_2 \leq \frac{\pi}{3} \\ &\theta_2 = \frac{\pi}{4} + \frac{\beta_2}{2} \end{aligned} \right\} \quad (\text{B.20})$$

where $\beta_2 = 2\theta_2 - \pi/2$, and $\theta_2 \simeq \arctan \left[\frac{5}{6} \sqrt{\frac{C_0}{T_0}} \right]$

B.2 Determining the failure envelope

In this section, another analytical procedure is carried out to derive the failure envelope within the tension - shear transition domain (segment P_1 to P_2) and the portion of the curve between P_2 and P_3 , for values of $\sigma \geq 0$, until $\sigma_3 = 0$. A key advantage of this approach is that the stress state can be computed for different values of the $(C_0/|T_0|)$ ratio. This contrasts with OGC, whose failure envelope corresponds to the specific case where $(C_0/|T_0|) = 8$, which thus constitutes a limitation that must be taken into account.

The abbreviated procedure for obtaining the failure envelope, described from this point on, is a full counterpart to the analytical development of the previous section.

As described by Úcar (2021), by applying the equations of equilibrium in a two-dimensional stress state and referring to Fig.B.2, it is known that:

$$\left\{ \begin{array}{l} \left[\sigma - \frac{1}{2} (\sigma_1 + \sigma_3) \right]^2 + \tau^2 = \left(\frac{\sigma_1 - \sigma_3}{2} \right)^2 \\ \text{Which is equivalent to writing:} \\ f(\sigma, \sigma_1, \sigma_3, \tau) = \left[\sigma - \frac{1}{2} (\sigma_1 + \sigma_3) \right]^2 + \tau^2 - \left(\frac{\sigma_1 - \sigma_3}{2} \right)^2 = 0 \end{array} \right. \quad (\text{B.21})$$

Considering that $\sigma_1 = \eta(\sigma_3)$, eq.B.21 can be rewritten as:

$$f(\sigma, \tau, \sigma_3) = 0 \quad (\text{B.22})$$

Moreover, if the family of plane curves defined by $f(\sigma, \tau, \sigma_3) = 0$ admits an envelope, then the functions $\sigma = \theta(\sigma_3)$ and $\tau = \varphi(\sigma_3)$ that define the parametric equations of this envelope satisfy the following system of equations:

$$\left. \begin{array}{l} f(\sigma, \tau, \sigma_3) = 0 \\ \frac{\partial f}{\partial \sigma_3} = 0 \end{array} \right\} \quad (\text{B.23})$$

According to eq.B.23, by expressing the stresses σ and τ as functions of σ_3 , we obtain the parametric equations that define the envelope.

Furthermore, if feasible, one may proceed by eliminating the stress σ_3 from the two equations given in eq.B.23, thereby deriving a relation of the form $\Omega(\sigma, \tau) = 0$, which also represents the envelope. This approach is valid in cases where the relationship between the principal stresses is linear, and as a result, the intrinsic failure envelope is a straight line.

It is also worth noting that the circles with variable radius σ_3 , described by eqs. B.21 and B.22, are referred to as involutes.

Taking the derivative of σ_1 with respect to σ_3 on both sides of eq. B.21, and rearranging terms:

$$2 \left[\sigma - \frac{1}{2} (\sigma_1 + \sigma_3) \right] \left[-\frac{1}{2} \left(\frac{\partial \sigma_1}{\partial \sigma_3} + 1 \right) \right] = \left(\frac{\sigma_1 - \sigma_3}{2} \right) \left(\frac{\partial \sigma_1}{\partial \sigma_3} - 1 \right) \quad (\text{B.24})$$

$$\left(\frac{\partial \sigma_1}{\partial \sigma_3} - 1 \right) \left(\frac{\sigma_1 - \sigma_3}{2} \right) + \left(\frac{\partial \sigma_1}{\partial \sigma_3} + 1 \right) \left[\sigma - \frac{1}{2} (\sigma_1 + \sigma_3) \right] = 0 \quad (\text{B.25})$$

By simplifying the above expression, the value of the normal stress σ becomes:

$$\sigma = \sigma_3 + \frac{(\sigma_1 - \sigma_3)}{\left(1 + \frac{\partial \sigma_1}{\partial \sigma_3} \right)} \quad (\text{B.26})$$

Substituting eq. B.26 into B.21 and solving for τ , we obtain:

$$\tau = \frac{\sigma_1 - \sigma_3}{\left(1 + \frac{\partial \sigma_1}{\partial \sigma_3} \right)} \sqrt{\frac{\partial \sigma_1}{\partial \sigma_3}} \quad (\text{B.27})$$

By dividing equation eq. B.27 by B.26, it is possible to write:

$$\frac{\tau}{(\sigma - \sigma_3)} = \left(\frac{\partial \sigma_1}{\partial \sigma_3} \right)^{1/2} \quad \therefore \quad \tau = (\sigma - \sigma_3) \left(\frac{\partial \sigma_1}{\partial \sigma_3} \right)^{1/2} \quad (\text{B.28})$$

From fig. B.2, it can be observed that $\tan 2\theta \cdot \tan \beta = -1$, and therefore, $(2\theta - \beta) = 90^\circ$. Under these conditions, and taking eq. B.28 into account, we obtain:

$$\left(\frac{\partial \sigma_1}{\partial \sigma_3} \right)^{1/2} = \frac{\tau}{(\sigma - \sigma_3)} = \tan \theta \quad (\text{B.29})$$

Moreover, as described by Úcar (2021), the slope of the plane curve $\sigma_1 = f(\sigma_3)$, which relates the principal stresses at failure, is given by the following equation:

$$\left(\frac{\partial \sigma_1}{\partial \sigma_3} \right) = \tan^2 \theta = \left(\frac{1 + \sin \beta}{1 - \sin \beta} \right) = \tan^2 \left(45^\circ + \frac{\beta}{2} \right) \quad (\text{B.30})$$

Substituting this expression into equation B.29 yields:

$$\tau = (\sigma - \sigma_3) \tan \overbrace{\left(45^\circ + \frac{\beta}{2} \right)}^{\theta} \quad (\text{B.31})$$

The above equation, when expressed in dimensionless form with respect to C_0 , becomes:

$$\frac{\tau}{C_0} = \left(\frac{\sigma}{C_0} - \frac{\sigma_3}{C_0} \right) \tan \left(45^\circ + \frac{\beta}{2} \right) \quad (\text{B.32})$$

For the sake of simplicity, the following notation is introduced:

$$y = \left(\frac{\tau}{C_0} \right), \quad x = \left(\frac{\sigma}{C_0} \right), \quad \text{and} \quad \bar{\sigma}_3 = \left(\frac{\sigma_3}{C_0} \right) \quad (\text{B.33})$$

All of this is intended to make eq.B.32 more practical to operate with, and independent of units of measurement. Thus, using the aforementioned notation, it can be expressed as:

$$y = (x - \bar{\sigma}_3) \tan \left(45^\circ + \frac{\beta}{2} \right) \quad (\text{B.34})$$

Solving for $\bar{\sigma}_3$ yields:

$$\bar{\sigma}_3 = x - y \tan \left(45^\circ - \frac{\beta}{2} \right) \quad (\text{B.35})$$

Which is equivalent to writing:

$$(\bar{\sigma}_3 - \xi) = (x - \xi) - y \tan \left(45^\circ - \frac{\beta}{2} \right) \quad (\text{B.36})$$

On the other hand, by equating eqs. B.5 and B.30, we obtain:

$$\frac{3a}{4} \cdot \frac{1}{(\bar{\sigma}_3 - \xi)^{1/4}} = \left(\frac{1 + \sin \beta}{1 - \sin \beta} \right) \quad (\text{B.37})$$

Solving for $(\bar{\sigma}_3 - \xi)$ from the above equation:

$$(\bar{\sigma}_3 - \xi) = \left(\frac{3a}{4} \right)^4 \left(\frac{1 - \sin \beta}{1 + \sin \beta} \right)^4 \quad (\text{B.38})$$

Substituting eq. B.38 into B.36, and introducing a new factor defined as:

$$\Omega = \left(\frac{3a}{4} \right)^4 \quad (\text{B.39})$$

We obtain:

$$\Omega \left(\frac{1 - \sin \beta}{1 + \sin \beta} \right)^4 = (x - \xi) - y \tan \left(45^\circ - \frac{\beta}{2} \right) \quad (\text{B.40})$$

By applying trigonometric transformations, eq.B.40 becomes:

$$\left. \begin{aligned} y &= (x - \xi) \left(\frac{\cos \beta}{1 - \sin \beta} \right) - \Omega \tan^7 \left(45^\circ - \frac{\beta}{2} \right) \\ \text{Which is also equivalent to:} \\ \left(\frac{\tau}{C_0} \right) &= (x - \xi) \tan \left(45^\circ + \frac{\beta}{2} \right) - \Omega \tan^7 \left(45^\circ - \frac{\beta}{2} \right) \end{aligned} \right\} \quad (\text{B.41})$$

For simplification of operations, the following change of variable is introduced:

$$X = (x - \xi) \quad (\text{B.42})$$

Thus, the final expression becomes:

$$y = X \left(\frac{\cos \beta}{1 - \sin \beta} \right) - \Omega \tan^7 \left(45^\circ - \frac{\beta}{2} \right) \quad (\text{B.43})$$

The next step consists in determining the slope of the envelope curve, given by $(dy/dx) = (dy/dX) = \tan \beta$. Under these conditions, by differentiating eq.B.43 and applying the necessary simplifications and trigonometric transformations, the following expression is obtained:

$$\begin{aligned} \left(\frac{dy}{dX} \right) = \tan \beta &= \left(\frac{\cos \beta}{1 - \sin \beta} \right) + X \left(\frac{1}{1 - \sin \beta} \right) \left(\frac{d\beta}{dX} \right) \\ &+ 7\Omega \tan^6 \left(45^\circ - \frac{\beta}{2} \right) \left(\frac{1}{1 - \sin \beta} \right) \left(\frac{d\beta}{dX} \right) \end{aligned} \quad (\text{B.44})$$

Which can be rearranged into the form:

$$\left[\tan \beta - \left(\frac{\cos \beta}{1 - \sin \beta} \right) \right] = \left(\frac{d\beta}{dX} \right) \left[\left(\frac{X}{1 - \sin \beta} \right) + \frac{7\Omega \tan^6 \left(45^\circ - \frac{\beta}{2} \right)}{1 + \sin \beta} \right] \quad (\text{B.45})$$

From the previous paragraphs, it is known that $x = (\sigma/C_0) = (X + \xi)$. Therefore, the objective is to express the normal stress as a function of the angle β , which defines the slope of the tangent to the envelope curve.

Through appropriate trigonometric operations, the preceding equation takes the following form:

$$\left(\frac{dX}{d\beta} \right) + \left(\frac{\cos \beta}{1 - \sin \beta} \right) X + \left[\frac{7\Omega \tan^8 \left(45^\circ - \frac{\beta}{2} \right)}{1 - \sin \beta} \cos \beta \right] = 0 \quad (\text{B.46})$$

The next step is to solve this first-order differential equation, which is linear since it is of the first degree in both the dependent variable X and its derivative.

In compact form, it becomes:

$$\left(\frac{dX}{d\beta}\right) + P(\beta) \cdot X + Q(\beta) = 0 \quad (\text{B.47})$$

From eq. B.46, it follows that:

$$\left. \begin{aligned} P(\beta) &= \left(\frac{\cos \beta}{1 - \sin \beta} \right) = \tan \left(\frac{\pi}{4} + \frac{\beta}{2} \right) \\ Q(\beta) &= \left[\frac{7\Omega \tan^8 \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \cos \beta}{1 - \sin \beta} \right] \end{aligned} \right\} \quad (\text{B.48})$$

The general solution of the first-order linear differential equation is:

$$X = e^{-\int P(\beta)d\beta} \left[\Omega_1 - \int e^{\int P(\beta)d\beta} \cdot Q(\beta)d\beta \right] \quad (\text{B.49})$$

Substituting $P(\beta)$ and $Q(\beta)$ into the above expression yields:

$$X = e^{-\int \frac{\cos \beta}{1 - \sin \beta} d\beta} \left(\Omega_1 - \int e^{\int \frac{\cos \beta}{1 - \sin \beta} d\beta} \cdot \left[\frac{7\Omega \tan^8 \left(\frac{\pi}{4} - \frac{\beta}{2} \right)}{1 - \sin \beta} \right] d\beta \right) \quad (\text{B.50})$$

Where Ω_1 is the integration constant.

Additionally, taking into account that:

$$\left. \begin{aligned} e^{-\int \frac{\cos \beta}{1 - \sin \beta} d\beta} &= (1 - \sin \beta) \\ e^{\int \frac{\cos \beta}{1 - \sin \beta} d\beta} &= \frac{1}{1 - \sin \beta} \end{aligned} \right\} \quad (\text{B.51})$$

Eq.B.50 then becomes:

$$X = (1 - \sin \beta) \left[\Omega_1 - 7\Omega \int \left(\frac{\tan^7 \left(\frac{\pi}{4} - \frac{\beta}{2} \right)}{1 - \sin \beta} \right) d\beta \right] \quad (\text{B.52})$$

Which is equivalent to:

$$X = (1 - \sin \beta) \left[\Omega_1 - 7\Omega \underbrace{\int \left(\frac{\tan^6 \left(\frac{\pi}{4} - \frac{\beta}{2} \right)}{\cos \beta} \right) d\beta}_I \right] \quad (\text{B.53})$$

The final step involves solving the integral, taking into account the change of variable $\tan(\beta/2) = t$, as it is convenient for integrating rational functions of $\sin \beta$ and $\cos \beta$.

$$I = f(\beta) = \int \frac{\tan^6 \left(\frac{\pi}{4} - \frac{\beta}{2} \right)}{\cos \beta} d\beta = -\frac{\tan^6 \left(\frac{\pi}{4} - \frac{\beta}{2} \right)}{6} \quad (\text{B.54})$$

Taking into account eqs. B.53 and B.54:

$$X = (1 - \sin \beta) \left(\Omega_1 + \frac{7\Omega}{6} \cdot \tan^6 \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \right) \quad (\text{B.55})$$

Considering, again, that $X = (x - \xi) = \left(\frac{\sigma}{C_0} - \xi \right)$, the normal stress as a function of β is given by:

$$\left. \begin{aligned} \bar{\sigma} - \xi &= (1 - \sin \beta) \left[\Omega_1 + \frac{7\Omega}{6} \cdot \tan^6 \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \right] \\ \bar{\sigma} &= \xi + (1 - \sin \beta) \left[\Omega_1 + \frac{7\Omega}{6} \cdot \tan^6 \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \right] \end{aligned} \right\} \quad (\text{B.56})$$

Expressing the last equation in a dimensionless form respect to $|T_0|$, we obtain

$$\frac{\sigma}{|T_0|} = \frac{\left(\xi + (1 - \sin \beta) \cdot \left[\Omega_1 + \frac{7\Omega}{6} \cdot \tan^6 \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \right] \right)}{|\xi|} \quad (\text{B.57})$$

For $\beta_1 \leq \beta \leq \frac{\pi}{2}$, when $\sigma_3 = 0$.

On the other hand, according to eq. B.41 (B.21), the shear strength is:

$$y = \bar{\tau} = \left[(\bar{\sigma} - \xi) - \Omega \left(\frac{1 - \sin \beta}{1 + \sin \beta} \right)^4 \right] \cdot \underbrace{\tan \left(\frac{\pi}{4} + \frac{\beta}{2} \right)}_{\left(\frac{\cos \beta}{1 - \sin \beta} \right)} \quad (\text{B.58})$$

By substituting the first equation of B.56 into the previous expression, we obtain:

$$\bar{\tau} = \left(\frac{\tau}{C_0} \right) = \left\{ \underbrace{(1 - \sin \beta) [\Omega_1 - 7\Omega I]}_{\bar{\sigma}} - \xi - \Omega \overbrace{\left(\frac{1 - \sin \beta}{1 + \sin \beta} \right)^4}^{\tan^8 \left(\frac{\pi}{4} - \frac{\beta}{2} \right)} \right\} \left(\frac{\cos \beta}{1 - \sin \beta} \right) \quad (\text{B.59})$$

Simplifying the equation, we obtain:

$$\bar{\tau} = \left(\frac{\tau}{C_0} \right) = \cos \beta \left\{ \left(\Omega_1 + \frac{7\Omega}{6} \cdot \tan^6 \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \right) - \Omega \frac{(1 - \sin \beta)^3}{(1 + \sin \beta)^4} \right\} \quad (\text{B.60})$$

Expressing the above equation in a dimensionless form respect to $|T_0|$, and considering trigonometric identities, we finally obtain:

$$\frac{\tau}{|T_0|} = \frac{\cos \beta \left\{ \Omega_1 + \Omega \cdot \tan^6 \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \cdot \left(\frac{7}{6} - \frac{1}{1 + \sin \beta} \right) \right\}}{|\xi|} \quad (\text{B.61})$$

Another important aspect worth highlighting refers to eqs. B.93 and B.94. These equations show that both the normal stress and the shear strength are represented parametrically as functions of the angle β .

Finally, by using eq. B.56, the integration constant Ω_1 can be determined by knowing the normal stress and the angle β in the particular case when the minor principal stress $\sigma_3 = 0$ and $\beta = \beta_1$.

As previously indicated in section B, when $\sigma_3 = 0$, it follows that $\sigma_1 = C_0 \Rightarrow \left(\bar{\sigma}_1 = \frac{\sigma_1}{C_0} = 1 \right)$. With these values, and by applying eq. B.9, the normal stress $\sigma = (\sigma)_{\sigma_3=0}$ as a function of the angle $\theta = \theta_1$ is given by:

$$\bar{\sigma} = \left(\frac{\sigma}{C_0} \right) = \left(\frac{\sigma_1}{C_0} \right) \cos^2 \theta_1 + \left(\frac{\sigma_3}{C_0} \right) \sin^2 \theta_1 = 1 \cdot \cos^2 \theta_1 \quad (\text{B.62})$$

On the other hand, when $\sigma_3 = 0$, eq. B.6 becomes:

$$\tan \theta_1 = \sqrt{\frac{3a}{4} \cdot \frac{1}{(-\xi)^{1/4}}} \quad (\text{B.63})$$

Therefore, the failure angle θ when $\sigma_3 = 0$ takes the value:

$$(\theta)_{\sigma_3=0} = \theta_1 = \arctan \left(\sqrt{\frac{3a}{4} \cdot \frac{1}{(-\xi)^{1/4}}} \right) \quad (\text{B.64})$$

Where the angle $\beta = \beta_1$

$$\theta_1 = \left(45^\circ + \frac{\beta_1}{2} \right) \quad (\text{B.65})$$

The integration constant Ω_1 is obtained from eq.B.56, considering that $\Omega = \left(\frac{3a}{4}\right)^4$, and using eqs.B.62 and B.65. Under these conditions, it follows that:

$$\Omega_1 = \frac{\cos^2 \theta_1 - \xi}{(1 - \sin \beta_1)} + 7I \left(\frac{3a}{4}\right)^4, \quad I = f(\beta_1) \quad (\text{B.66})$$

On the other hand, knowing that $\cos^2 \theta_1 = 0.5(1 - \sin \beta_1)$, and considering $I = f(\beta_1)$ from eq.B.54, the above equation expressed as a function of the angle β_1 becomes:

$$\Omega_1 = \left[\frac{1}{2} - \frac{\xi}{(1 - \sin \beta_1)} \right] - \frac{7}{6} \cdot \left(\frac{3a}{4}\right)^4 \cdot \tan^6 \left(\frac{\pi}{4} - \frac{\beta_1}{2} \right) \quad (\text{B.67})$$

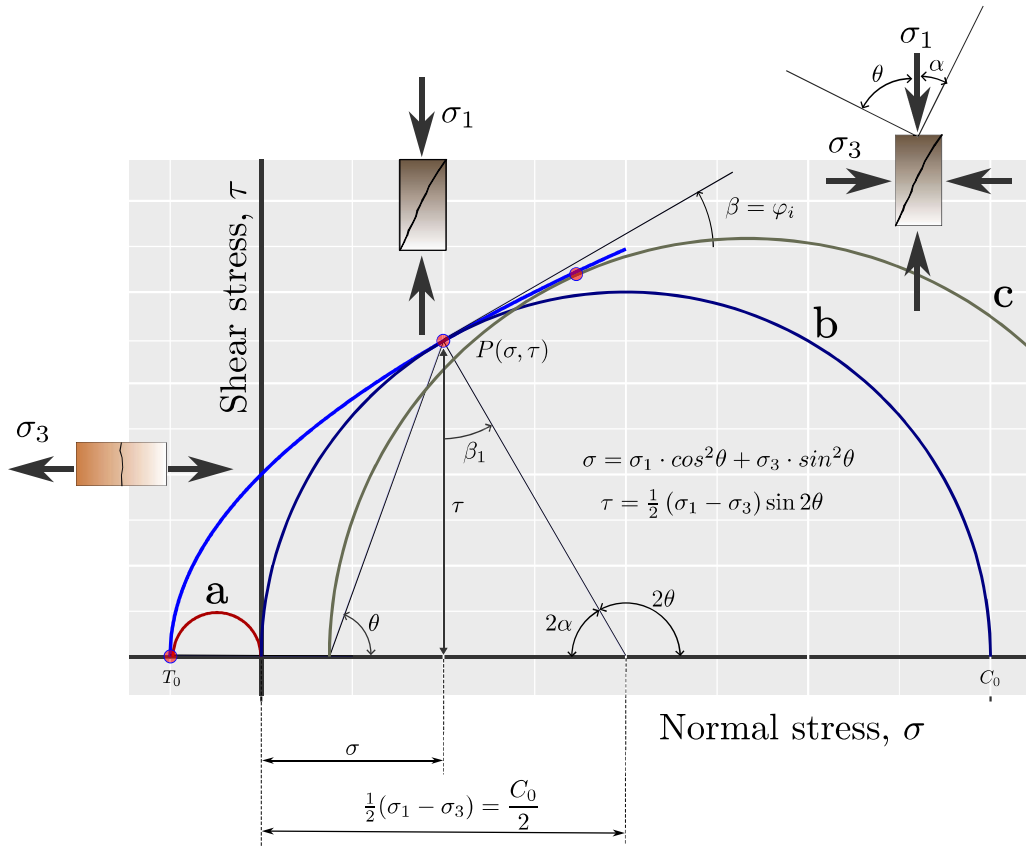


Figure B.2: Boundary conditions of failure: a) Pure traction test ($\sigma_1 = 0, \sigma_3 = T_0$). b) Uniaxial compressive test ($\sigma_1 = C_0, \sigma_3 = 0$). c) Generic triaxial test ($\sigma_1 \geq \sigma_3 \geq 0$). Values of shear, τ , and normal, σ stresses as a function of (σ_1, σ_3 , and θ). For the case of the uniaxial compressive test, main angles are indicated. Additionally, at Point P of the failure envelope, it can be observed that $\tan 2\theta \cdot \tan \beta = -1$, therefore, $(2\theta - \beta) = 90^\circ$ and $\theta = (45^\circ + \beta/2)$.

B.3 Alternative procedure

The following procedure allows for the reduction of a series of steps in the analytical method described in the previous section, B.1. As previously discussed in detail in the preceding sections,

in the new analytical method referred to as NGC, the parameters involved are obtained through a set of boundary conditions, via Equations B.58 and B.60.

Furthermore, on the basis of these parameters, the pure tensile stress state and the hybrid or transitional zone—comprising combined tensile and shear stresses—have been determined, both in the (σ_3, σ_1) and the (σ, τ) spaces.

A system of five non-linear equations was solved by means of well-established numerical methods and their associated algorithms, taking as the main governing parameter the ratio C_0/T_0 or its reciprocal.

On the other hand, this alternative method significantly simplifies the calculations with respect to the rigorous analytical approach, by obtaining the parameter a through a linear regression using a power-law fit, and λ through a simple linear relationship (see Figure 4 and Equations 21 and 22 of the main text).

At the same time, the major principal stress $\sigma_1 = f(a, \lambda, \sigma_3)$ is determined, together with the angles β_1 and β_2 , as well as the normal and shear stresses acting on the failure plane.

From previous developments, it is known that σ and τ can be expressed as:

$$\begin{aligned}\sigma &= \sigma_1 \cos^2 \theta + \sigma_3 \sin^2 \theta \\ \tau &= \frac{1}{2} (\sigma_1 - \sigma_3) \sin 2\theta\end{aligned}\tag{B.68}$$

And, in dimensionless form with respect $|T_0|$:

$$\left. \begin{aligned}\frac{\sigma}{|T_0|} &= \frac{\sigma_1}{|T_0|} \cos^2 \theta + \frac{\sigma_3}{|T_0|} \sin^2 \theta \\ \frac{\tau}{|T_0|} &= \frac{1}{2} \left(\frac{\sigma_1}{|T_0|} - \frac{\sigma_3}{|T_0|} \right) \sin 2\theta\end{aligned}\right\}\tag{B.69}$$

On the other hand: $\bar{\sigma}_1 = a(-\xi)^{3/4} + \lambda|\xi|$, and remembering that $\bar{\sigma}_1 = \frac{\sigma_1}{c_0}$

$$\begin{aligned}\bar{\sigma}_1 &= a(\bar{\sigma}_3 - \xi)^{3/4} + \lambda|\xi| \Rightarrow \frac{\sigma_1}{|T_0|} = \frac{\overbrace{a(\bar{\sigma}_3 - \xi)^{3/4} + \lambda|\xi|}^{\bar{\sigma}_1}}{|\xi|} \\ \frac{\sigma_1}{|T_0|} &= \frac{\bar{\sigma}_1}{|\xi|} \quad , \quad \frac{\sigma_3}{|T_0|} = \frac{\bar{\sigma}_3}{|\xi|} \quad \text{where } \xi \leq \overbrace{\left(\frac{\sigma_3}{C_0}\right)}^{\bar{\sigma}_3}\end{aligned}\tag{B.70}$$

And transforming the previous eqs.:

$$\left. \begin{aligned}\frac{\sigma}{|T_0|} &= \frac{1}{2} \left[\left(\frac{\sigma_1}{|T_0|} + \frac{\sigma_3}{|T_0|} \right) - \sin \beta \left(\frac{\sigma_1}{|T_0|} - \frac{\sigma_3}{|T_0|} \right) \right] \\ \frac{\tau}{|T_0|} &= \frac{1}{2} \left(\frac{\sigma_1}{|T_0|} - \frac{\sigma_3}{|T_0|} \right) \cos \beta\end{aligned}\right\}\tag{B.71}$$

The angle β ranges from $\beta = 90^\circ$ ($\sigma = T_0$) to $\beta = (\beta_1)$ when $\sigma_3 = 0$, that is, $\sigma =$

$C_0 \left(\frac{1 - \sin \beta_1}{2} \right)$. This is equivalent to writing: $\xi \leq \bar{\sigma} \leq \left(\frac{1 - \sin \beta_1}{2} \right)$, and with respect to T_0 : $T_0 \leq \sigma \leq C_0 \cdot \left(\frac{1 - \sin \beta_1}{2} \right)$.

Additionally, for different values of the minor principal stress, the angle β is obtained through the equation:

$$\theta = \arctan \left(\sqrt{\frac{3a}{4(\bar{\sigma}_3 - \xi)^{1/4}}} \right), \beta = (2\theta - 90) \quad (\text{B.72})$$

It should be noted that the value $\beta = \beta_2$ corresponds to the condition $\sigma = 0$, and is obtained together with β_1 (for $\sigma_3 = 0$) by solving the system of five non-linear equations.

Furthermore, it has been demonstrated that, for different values of $C_0/|T_0|$, the minimum principal stress lies approximately at $\sigma_3 = (7/8) \cdot T_0$, that is:

$$\bar{\sigma}_3 = \frac{7}{8} \cdot \left(\frac{T_0}{C_0} \right) = \frac{7}{8} \cdot \xi \quad (\text{B.73})$$

In other words, it remains practically constant within the closed interval $6 \leq (C_0/|T_0|) \leq 24$, thereby avoiding laborious calculations and consequently leading to a significant saving of time when applying the aforementioned alternative procedure.

Continuing with the computational development, by substituting Eq. B.73 into Eq. B.72 and simplifying, the angles θ and β are obtained for $\sigma = 0$.

$$\theta = \theta_2 = \arctan \left(\frac{1}{2} \sqrt{\frac{5a}{|\xi|^{1/4}}} \right), \quad \beta_2 = (2\theta_2 - 90) \quad (\text{B.74})$$

On the other hand, according to the NGC, the major principal stress is given by the following equation:

$$\bar{\sigma}_1 = a(\bar{\sigma}_3 - \xi)^{3/4} + \lambda|\xi| \quad (\text{B.75})$$

From the previous sections, it is known that, upon applying the curve-fitting technique based on linear regression, the parameters a and λ are given by:

$$a = 0.531 \left(\frac{C_0}{|T_0|} \right)^{3/4} \quad (\text{B.76})$$

and

$$\lambda = 0.472 \left(\frac{C_0}{|T_0|} \right) \quad (\text{B.77})$$

On the other hand, through equations B.76 and B.77, it can be observed that $a \simeq \lambda^{3/4}$.

Finally, it should be noted that, in the next section, a parameter of fundamental importance is determined as a complementary calculation. Its value represents the shear strength at zero normal stress, that is, $(\tau)_{\sigma=0} = \tau_0$, which also depends on the ratio $C_0/|T_0|$, as demonstrated below.

B.4 Particular cases

B.4.1 The case of $\sigma = 0$

In light of the above, the next step is based on applying Equation B.68 and considering the effective normal stress $\bar{\sigma} = 0$. At the same time, the major principal stress σ_1 , together with the shear stress, is expressed in dimensionless form with respect to C_0 . Under these conditions, the stresses take the following form:

$$\bar{\sigma}_1 = \left(\frac{\sigma_1}{C_0} \right) = -\bar{\sigma}_3 \tan^2 \theta_2 \quad (\text{B.78})$$

$$(\bar{\tau})_{\sigma=0} = \left(\frac{\tau}{C_0} \right)_{\sigma=0} = \bar{\tau}_0 = \frac{1}{2} (\bar{\sigma}_1 - \bar{\sigma}_3) \sin 2\theta_2 = -\bar{\sigma}_3 \tan \theta_2 \quad (\text{B.79})$$

$$\text{where } \bar{\sigma}_3 = \left(\frac{\sigma_3}{C_0} \right) \text{ and } \theta_2 = \left(45^\circ + \frac{\beta_2}{2} \right)$$

It can also be observed that the value $\beta = \beta_2$ corresponds to the condition $\bar{\sigma} = 0$.

Moreover, it is worth emphasising that, on the basis of hundreds of laboratory tests (references to be included), the values of C_0 and T_0 have been obtained. The ratio $\xi = (T_0/C_0)$, or its reciprocal $\frac{C_0}{T_0}$, thus emerges as the most relevant parameter within the structure and execution of the computational programme, and consequently in the solution of the system composed of five non-linear equations. In addition, it is demonstrated within this mathematical framework and its associated algorithms that the minor principal stress tends towards $\sigma_3 \approx (7/8)T_0$ as the normal stress approaches zero. Therefore,

$$\bar{\sigma}_3 = \frac{\sigma_3}{C_0} = \frac{7}{8} \left(\frac{T_0}{C_0} \right) = \frac{7}{8} \xi \quad (\text{B.80})$$

In other words, this value remains practically constant within the interval analysed, namely $6 \leq \left(\frac{C_0}{|T_0|} \right) \leq 30$, thereby avoiding cumbersome calculations and allowing for a substantial saving in computational time, while still achieving a high level of accuracy.

As an additional point, when applying the Úcar criterion to the particular case in which $\bar{\sigma} = 0$, it is obtained—through the solution of a cubic equation—that $\bar{\sigma}_3$ is also constant within the same interval, taking the value $\bar{\sigma}_3 = (2/3)\xi$, that is, $\sigma_3 = (2/3)T_0$. This indicates that the failure envelope in the tensile domain does not represent the actual stress field, since σ_3 is located further away from T_0 .

On the other hand, as previously noted, when applying the Griffith criterion it is known that $\sigma_3 = 2(\sqrt{2} - 1)T_0 = 0.83 \cdot T_0$, a value that is slightly further from that obtained using the newly proposed criterion ($\sigma_3 = 7/8 \cdot T_0$).

Resuming the derivation and taking into account Equation B.72, together with the condition $\bar{\sigma}_3 = \frac{\sigma_3}{C_0} = \frac{7}{8} \left(\frac{T_0}{C_0} \right) = \frac{7}{8} \xi$, the following expression is obtained:

$$\tan^2 \theta_2 = \left(\frac{3a}{4} \right) \frac{1}{\left(\frac{1}{8} |\xi| \right)^{1/4}} \quad (\text{B.81})$$

Hence, the value of $\tan \theta_2$ for the case $\bar{\sigma} = 0$ is given by

$$\tan \theta_2 = \left(\frac{\sqrt{3a}}{2} \right) (0.125 \cdot |\xi|)^{-1/8} \quad (\text{B.82})$$

Replacing the value of (a) indicated in B. 76, the last equation becomes

$$\tan \theta_2 = 0.817 \sqrt{\frac{C_0}{|T_0|}} \simeq \frac{5}{6} \sqrt{\frac{C_0}{|T_0|}} \quad (\text{B.83})$$

Substituting the value of $\tan \theta_2$ into Equation B.74 yields:

$$\bar{\tau}_{(\sigma_3=0)} = -\bar{\sigma}_3 \tan \theta_2 = \left(\frac{7}{8} |\xi| \right) 0.817 \left(\frac{C_0}{|T_0|} \right)^{1/2} \quad (\text{B.84})$$

$$\bar{\tau}_{(\sigma=0)} = \left(\frac{\tau}{C_0} \right) = 0.714 |\xi| \left(\frac{C_0}{|T_0|} \right)^{1/2} \quad (\text{B.85})$$

$$\tau_{(\sigma=0)} = |T_0| \cdot 0.714 \left(\frac{C_0}{|T_0|} \right)^{-1/2} \quad (\text{B.86})$$

Dividing both sides of the previous equation by $|T_0|$, one finally obtains:

$$\left(\frac{\tau}{|T_0|} \right) = 0.714 \left(\frac{C_0}{|T_0|} \right)^{-1/2} \quad \left(\frac{\tau}{|T_0|} \right) = \frac{5}{7} \sqrt{\frac{C_0}{|T_0|}} \quad (\text{B.87})$$

If we examine the case of $\frac{C_0}{|T_0|} = 8$, which is the only ratio admitted by OGC:

$$\left(\frac{\tau}{|T_0|} \right) = \frac{5}{7} \sqrt{\frac{C_0}{|T_0|}} = \frac{5}{7} \sqrt{8} = \frac{5}{7} \cdot 2.828 = 2.02 \quad (\text{B.88})$$

There is a complete concordance with the OGC prediction for $\tau_{(\sigma=0)} = 2$.

Figure B.3 shows the variation of $\tau_{(\sigma=0)}$ with the ratio $\frac{C_0}{|T_0|}$ according to eq. B.88. Notice how, for $\frac{C_0}{|T_0|} = 8$, $\tau_{(\sigma=0)} = 2|T_0|$ as was the case for the OGC (see section A).

In a similar vein, we can plot (using eq. B.72) the values of β_1 (that corresponds to $\sigma_3 = 0$) and β_2 (when $\sigma = 0$) against the ratio $\frac{C_0}{|T_0|}$. Figure B.4 shows the result of such comparison. As

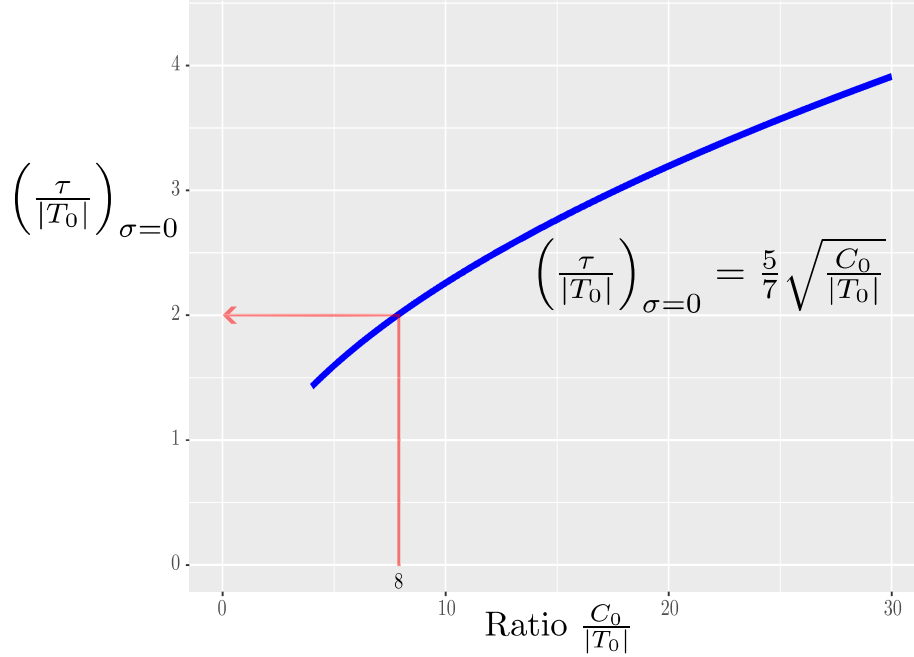


Figure B.3: Variation of $\tau_{(\sigma=0)}$ with the ratio $\frac{C_0}{|T_0|}$. For the ratio $\frac{C_0}{|T_0|} = 8$, that we saw in section A, corresponds to the OGC, $\tau_{(\sigma=0)} = 2|T_0|$ as was the case for the OGC.

discussed in the paper, together with fig.B.3, this demonstrates that, for all practical purposes, OGC constitutes a specific case of NGC when $C_0/|T_0| = 8$. This, in turn, suggests that NGC represents a natural extension of OGC, whereby the fixed ratio $C_0/|T_0| = 8$ is abandoned in favour of variable ratios supported by experimental evidence.

When applying Eq. B.87 derived from the NGC, it is noteworthy that the shear strength at $\sigma = 0$ is $\tau = 2|T_0|$. This result is consistent with the OGC, for which $\frac{C_0}{T_0} = 8$.

The linear Mohr–Coulomb criterion was later reformulated as $\tau = 2|T_0| + \sigma \tan \varphi$, commonly referred to as the Modified Griffith Criterion (Gudmundsson, 2011). This formulation allows the compressional stress regime to be linked with the Griffith failure parabola in the tensile domain.

However, the relationship $C = 2|T_0|$ holds only when the internal friction angle is $\varphi = 36.9^\circ$. Under the Mohr–Coulomb framework, $C_0 = 2c \tan(45^\circ + \varphi/2)$; therefore, assuming $C = 2|T_0|$ leads to $C_0 = 4|T_0| \tan(45^\circ + \varphi/2)$. This expression yields $C_0 = 8|T_0|$ when $\varphi = 36.9^\circ$, whereas application of the original Griffith criterion (OGC) implies $\varphi = 30^\circ$.

Finally, Table B.2 shows the results of applying the different approaches to the obtention of the relevant parameters for a specific case: a rock with $C_0 = 25$ MPa, $T_0 = -2.5$ MPa and therefore, $\xi = -0.1$ and $C_0/|T_0| = 10$. The equations of the analytical approach are implemented in the R script of section C. The equations obtained in this section for the alternative procedure were implemented with a handheld calculator.

B.4.2 The case of $\sigma_3 = 0$

As a final step, the next task is to determine β when the minor principal stress $\sigma_3 = 0$. Under these conditions, and considering that $\theta_{(\sigma_3=0)} = \theta_1$, Eq. (B.6) becomes:

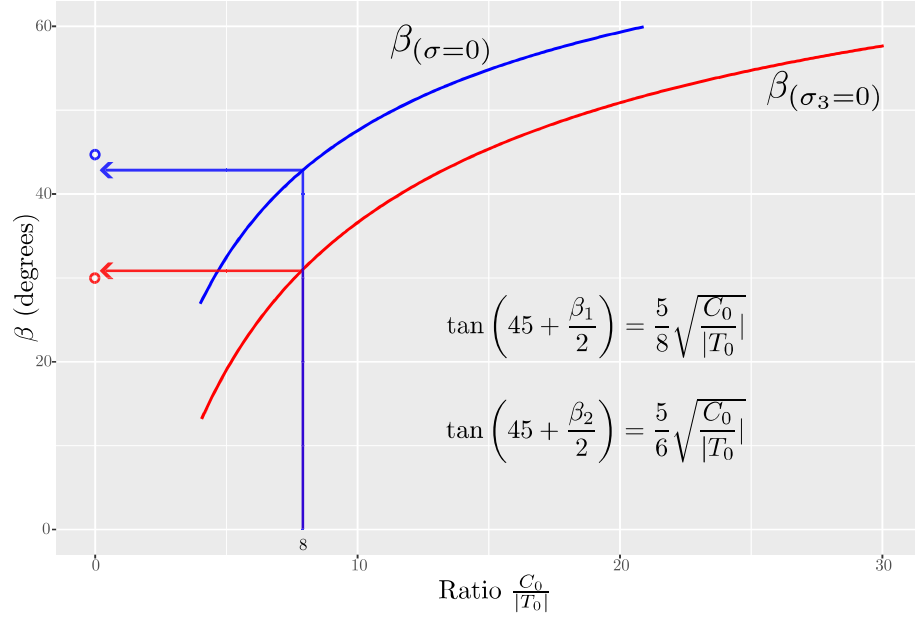


Figure B.4: Variation on β_1 (β when $\sigma_3 = 0$) and β_2 (that corresponds to β when $\sigma = 0$) with the Ratio $\frac{C_0}{|T_0|}$. Notice how, for $\frac{C_0}{|T_0|} = 8$ the obtained values are close to those of the OCG (blank circles at 45° and 30° , respectively).

	Rigorous	Alternative
a	2.97	2.99
λ	4.72	4.72
β_1	36.64°	36.76°
β_2	47.63°	47.56°
$\sigma_{1\sigma=0}$ (MPa)	14.57	14.45
$\sigma_{3\sigma=0}$ (MPa)	-2.19	-2.19

Table B.2: Values, for a rock with $C_0 = 25$ MPa, $T_0 = -2.5$ MPa and therefore, $C_0/|T_0| = 10$, obtained with the two procedures included in this work (with the abacus we can obtain the values of a and λ and, from that point, the rest of parametres can be obtained either via the rigorous or the alternative method).

$$\tan^2 \theta_1 = \left(\frac{3a}{4} \right) \cdot \frac{1}{(-\xi)^{1/4}} \quad (\text{B.89})$$

Substituting the value of a from Eq. (B.16) yields:

$$\tan^2 \theta_1 = \frac{3}{4} \cdot \left[0.53 \cdot \left(\frac{C_0}{|T_0|} \right) \right]^{3/4} \cdot \frac{1}{\left(\frac{|T_0|}{C_0} \right)^{1/4}} \quad (\text{B.90})$$

After simplification, this yields:

$$\tan \theta_1 = \frac{5}{8} \sqrt{\frac{C_0}{|T_0|}} \quad (\text{B.91})$$

Considering that $\theta_1 = \left(45^\circ + \frac{\beta_1}{2} \right)$, the value of $\beta_{1(\sigma_3=0)}$ is therefore obtained as $\beta_{1(\sigma_3=0)} = 90^\circ - 2\theta_1$

In addition, Eqs. (B.83) and (B.91) yield an important relationship between the fracture angles θ_2 and θ_1 , from which it follows, to a good approximation, that:

$$\frac{\tan \theta_2}{\tan \theta_1} = \frac{4}{3} \quad (\text{B.92})$$

Table B.3 summarises the equations required to compute the parameters a , λ , σ , τ , θ_1 , β_1 , θ_2 , and β_2 , considering stress states within the interval $\beta_1 \leq \beta \leq 90^\circ$.

To complete the summary of table B.3: for the rigorous method the needed eqs. are:

$$\frac{\sigma}{|T_0|} = \frac{\left(\xi + (1 - \sin \beta) \cdot \left[\Omega_1 + \frac{7\Omega}{6} \cdot \tan^6 \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \right] \right)}{|\xi|} \quad (\text{B.93})$$

$$\frac{\tau}{|T_0|} = \frac{\cos \beta \left\{ \Omega_1 + \Omega \cdot \tan^6 \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \cdot \left(\frac{7}{6} - \frac{1}{1 + \sin \beta} \right) \right\}}{|\xi|} \quad (\text{B.94})$$

and

$$\Omega_1 = \left[\frac{1}{2} - \frac{\xi}{(1 - \sin \beta_1)} \right] - \frac{7}{6} \cdot \left(\frac{3a}{4} \right)^4 \cdot \tan^6 \left(\frac{\pi}{4} - \frac{\beta_1}{2} \right) \quad (\text{B.95})$$

NGC's MAIN EQUATIONS		
(σ, τ) space	$\frac{\sigma}{ T_0 } = \frac{1}{2} \left[\left(\frac{\sigma_1}{ T_0 } + \frac{\sigma_3}{ T_0 } \right) - \sin \beta \left(\frac{\sigma_1}{ T_0 } - \frac{\sigma_3}{ T_0 } \right) \right]$	$\frac{\tau}{ T_0 } = \frac{1}{2} \left(\frac{\sigma_1}{ T_0 } - \frac{\sigma_3}{ T_0 } \right) \cos \beta$
(σ_3, σ_1) space	$\bar{\sigma}_1 = a (\bar{\sigma}_3 - \xi)^{3/4} + \lambda \xi $ where:	$\xi = \left(\frac{T_0}{C_0} \right)$
	$\sigma_1 = a \cdot C_0 \cdot \left(\frac{\sigma_3}{C_0} - \xi \right)^{3/4} + \lambda T_0 $	$a = 0.531 \left(\frac{C_0}{ T_0 } \right)^{3/4}$
		$\lambda = 0.472 \cdot \left(\frac{C_0}{ T_0 } \right)$
PARTICULAR CASES		
if $\sigma = 0$	$\theta = \theta_2$	$\tan(\theta_2) = \frac{5}{6} \sqrt{\frac{C_0}{ T_0 }}$
		$\theta_2 = (45^\circ + \beta_2/2)$
if $\sigma_3 = T_0$	$\sigma_1 = \lambda \cdot T_0$	
if $\sigma_3 = 0$	$\theta = \theta_1$	$\tan(\theta_1) = \frac{5}{8} \sqrt{\frac{C_0}{ T_0 }}$
		$\theta_1 = (45^\circ + \beta_1/2)$
$\beta = 90^\circ$	$\sigma = T_0$	$\tau = 0$
$\beta = \beta_1$	$\sigma_3 = 0$	$\sigma_1 = \frac{C_0}{2} (1 - \sin \beta_1)$
$\beta = \beta_2$	$\sigma = 0$	$\sigma_3 = \frac{7}{8} T_0$
		$\frac{\tau}{ T_0 } = \frac{5}{7} \sqrt{\frac{C_0}{ T_0 }}$
if $\beta_2 > \beta_1$		$\frac{\tan \theta_2}{\tan \theta_1} = \frac{4}{3}$

Table B.3: Main equations of the NGC for obtaining the strength curves in both the (σ, τ) and the (σ_3, σ_1) spaces, and main particular cases of the NGC. Note that $\bar{\sigma}_1 = \sigma_1/C_0$ and $\bar{\sigma}_3 = \sigma_3/C_0$.

C R Scripts to reproduce calculations and figures

C.1 Script for generating NGC plus UC curves

This script starts from known values of C_0 and T_0 , from which it derives the relevant parameters for both the UC and NGC failure criteria. It then generates dense sequences of values for σ , τ , β , σ_1 , and σ_3 (with the density of the sequences being user-defined), and exports corresponding data tables. These data frames are subsequently used to produce (σ, τ) and (σ_1, σ_3) plots using ggplot2. There is some clutter within the code, heritage of previous versions and functions, but mostly it does perform as advertised.

The full script used to compute the parameters and generate the figures is openly accessible on GitHub at: [NGC-UC.R](#).

```

1 #####
2 ##### New Griffith (1924) plus Ucar (2021) criterion
3 ##### with manual entry of parameters C0 and T0,
4 ##### plots sigma/tau and sigma3/sigma1
5 ##### optional production of plots from combined table
6 ##### v9
7 #####
8
9 # 1. Install and load libraries
10
11 paquetes <- c("ggplot2", "dplyr")
12 instalar_faltantes <- paquetes[!sapply(paquetes, requireNamespace,
13   quietly = TRUE)]
14 if (length(instalar_faltantes) > 0) install.packages(instalar_faltantes)
15 lapply(paquetes, library, character.only = TRUE)
16
17 # 2. Intro parameters C0 and T0
18 # Obtained from lab test, regression from triaxial data, etc
19 sigma_t_adj <- -1
20 sigma_c_adj <- 10
21 xi_adj <- sigma_t_adj / sigma_c_adj
22
23
24 # 3. Obtention of Ucar's curve
25
26 # 3.1. Obtaining k1 and k2
27 k1 <- (-(1 + abs(xi_adj)) + sqrt((1 + 7 * abs(xi_adj)) * (1 - abs(xi_adj)
28   ))) / (2 * abs(xi_adj))
29 k2 <- (1 - k1 * (-xi_adj)) / sqrt(pmax(0.0001, -xi_adj))
30
31 # 3.2. Defining a dense sequence for sigma3 (0-> 500 MPa)
32 # (edit range if needed)
33 # and the dimensionless version
34
35 sigma_3 <- seq(0, 500, length.out = 1000) # MPa
36 sigma3_bar <- sigma_3 / sigma_c_adj
37
38
39 # 3.3. sigma_1 (bar), alpha (rad), tau and sigma_n
40

```

```

41 # UC with normalized sigma_1
42 sigma1_bar <- k1 * (sigma3_bar - xi_adj) + k2 * sqrt(pmax(0, sigma3_bar
   - xi_adj))
43
44 # Alpha (rad)
45 tan_alpha2_ucar <- k1 + (k2 / (2 * sqrt(pmax(0.0001, sigma3_bar - xi_adj
   ))))
46 alpha_rad <- atan(sqrt(tan_alpha2_ucar)) # en radianes
47
48 # Normalized sigma_n and tau
49 sigma_n_bar <- sigma1_bar * cos(alpha_rad)^2 + sigma3_bar * sin(alpha_
   rad)^2
50 tau_bar <- 0.5 * (sigma1_bar - sigma3_bar) * sin(2 * alpha_rad)
51
52 # 3.4. Back to MPa
53
54 sigma_1 <- sigma1_bar * sigma_c_adj
55 sigma_n <- sigma_n_bar * sigma_c_adj
56 tau <- tau_bar * sigma_c_adj
57 alpha_deg <- alpha_rad * 180 / pi
58 beta <- 2*alpha_deg - 90
59
60
61
62 # 4. Creating UC data.frame and exporting a csv file
63
64 curve_ucar_theta <- data.frame(
65   sigma_n = sigma_n,
66   tau = tau,
67   alpha =alpha_deg,
68   sigma_1 = sigma_1,
69   sigma_3 = sigma_3
70 )
71
72
73 # 4.1. To CSV
74
75 write.csv(curve_ucar_theta, "curva_UC_original.csv", row.names = FALSE)
76
77
78 # 5. NGC
79
80 # 5.1. beta1 directly from a
81
82 calcular_beta1 <- function(a, ratio) {
83   arg <- 0.75 * a * (-ratio)^(-0.25)
84   if (!is.finite(arg) || arg < 0) return(NA_real_)
85   beta1_NGC <- 2 * (atan(sqrt(arg)) - pi/4)
86   return(beta1_NGC)
87 }
88
89
90 # 5.2. Function to obtain NGC parameters, known beta1
91 ecuacion_objetivo <- function(a, ratio, xi_adj) {
92   if (!is.finite(a) || a <= 0) return(1e12)
93
94   lambda <- tryCatch((1 - a * (-ratio)^0.75) / (-ratio), error =
     function(e) return(1e12))

```

```

95   if (!is.finite(lambda)) return(1e12)
96
97   beta_1_NGC <- calcular_beta1(a, ratio)
98   if (is.na(beta_1_NGC)) return(1e12)
99
100  arg2 <- 1 - (xi_adj/((xi_adj/(1-sin(beta_1_NGC)))-(0.5-1/exp(1.008))))
101  beta_2 <- asin(arg2)
102  tan_term <- tan(pi / 4 + 0.5 * beta_2)
103  sigma_3 <- tryCatch((tan_term^2 / (0.75 * a))^(-4) + ratio, error =
104    function(e) return(1e12))
105  if (!is.finite(sigma_3)) return(1e12)
106
107  term1 <- 3 * a * sigma_3
108  term2 <- 4 * a * (sigma_3 - ratio)
109  term3 <- 4 * (sigma_3 - ratio)^0.25 * lambda * abs(ratio)
110  residuo <- term1 + term2 + term3
111  if (!is.finite(residuo)) return(1e12)
112
113  penalizacion <- 0
114  if (sigma_3 > 0) penalizacion <- penalizacion + 1e6 * sigma_3
115  if (sigma_3 < xi_adj) penalizacion <- penalizacion + 1e6 * abs(sigma_3
116    - xi_adj)
117
118  return(residuo^2 + penalizacion)
119 }
120
121 calcular_parametros_NGC <- function(xi_adj, sigma_c_adj, sigma_t_adj) {
122   ratio <- sigma_t_adj / sigma_c_adj
123
124   opt <- optim(
125     par = 5,
126     fn = ecuacion_objetivo,
127     method = "L-BFGS-B",
128     lower = 2, upper = 10,
129     ratio = ratio,
130     xi_adj = xi_adj
131   )
132
133   a <- opt$par
134   lambda <- (1 - a * (-ratio)^0.75) / (-ratio)
135   beta_1_NGC <- calcular_beta1(a, ratio)
136   arg2 <- 1 - (xi_adj/((xi_adj/(1-sin(beta_1_NGC)))-(0.5-1/exp(1.008))))
137   beta_2 <- asin(arg2)
138   tan_term <- tan(pi / 4 + 0.5 * beta_2)
139   sigma_3 <- (tan_term^2 / (0.75 * a))^(-4) + ratio
140
141   return(list(
142     a = a,
143     lambda = lambda,
144     beta_1_NGC = beta_1_NGC,
145     beta_2_NGC = beta_2,
146     sigma_3_NGC = sigma_3,
147     residuo = opt$value
148   ))
149 }
150
151 resultado <- calcular_parametros_NGC(xi_adj, sigma_c_adj, sigma_t_adj)

```

```

151 print(resultado)
152
153
154 # 5.3. Global environment extraction
155 list2env(resultado, envir = .GlobalEnv)
156
157 beta_2_NGC_grad<-beta_2_NGC*180/pi
158
159 # beta_1 in degrees
160 beta_1_grad <- 2 * (atan(sqrt(0.75 * a * (-xi_adj)^(-0.25))) * 180 / pi)
161   - 90
162
163 # Dense sequence of beta_grad from 90 to beta_1_grad
164 beta_grad_vals <- seq(90, beta_1_grad, length.out = 200)
165
166 # 6. Creating the data.frame curve NGC
167 curve_NGC <- data.frame(beta_grad = beta_grad_vals) %>%
168   mutate(
169     beta_rad = beta_grad * pi / 180,
170     alpha_NGC=45+0.5*beta_grad_vals,
171     tan_term = tan((beta_grad + 90) / 2 * pi / 180),
172     sigma_3_barNGC = xi_adj + (0.75 * a / tan_term^2)^4,
173     sigma_3_NGC = sigma_3_barNGC * sigma_c_adj,
174     sigma_3_priNGC = sigma_3_NGC / abs(sigma_t_adj),
175     sigma_1_barNGC = a * (sigma_3_barNGC - xi_adj)^0.75 + lambda * (-xi_
176       adj),
177     sigma_1_NGC = sigma_1_barNGC * sigma_c_adj,
178     sigma_1_priNGC = sigma_1_NGC / abs(sigma_t_adj),
179     sigma_n_priNGC = 0.5 * ((sigma_1_priNGC + sigma_3_priNGC) -
180       (sigma_1_priNGC - sigma_3_priNGC) * sin(
181         beta_rad)),
182     tau_pri_NGC = 0.5 * (sigma_1_priNGC - sigma_3_priNGC) * cos(beta_rad
183       ),
184     tau_NGC = tau_pri_NGC * abs(sigma_t_adj),
185     sigma_n_NGC = sigma_n_priNGC * abs(sigma_t_adj)
186   ) %>%
187   select(sigma_n_NGC, tau_NGC,alpha_NGC
188     , sigma_1_NGC, sigma_3_NGC)
189
190 # Exporting as table in CSV format
191 write.csv(curve_NGC, "curva_NGC_beta.csv", row.names = FALSE)
192
193 # 7. Plot NGC plus UC in tau-sigma space
194 ggplot(curve_NGC, aes(x = sigma_n_NGC, y = tau_NGC)) +
195   geom_line(color = "blue", size = 1.2) +
196   geom_line(data = curve_ucar_theta, aes(x = sigma_n, y = tau),
197     color = "orange", size = 1.2, linetype = "solid", alpha=0.7)
198   +
199   labs(
200     title = "Curva de rotura NGC",
201     x = expression(sigma[n]~(MPa)),
202     y = expression(tau~(MPa))
203   ) +
204   xlim(-2,11)+ # edit limits to adjust plot IMPORTANT
205   ylim(0,8)+ # edit limits to adjust plot IMPORTANT
206   coord_fixed(ratio = 1) +
207   theme_gray()

```

```

204
205 # 8. Plot NGC plus UC in sigma3-sigma1 space
206 ggplot(curve_NGC, aes(x = sigma_3_NGC, y = sigma_1_NGC)) +
207   geom_line(color = "blue", size = 1.2, alpha=0.7) +
208   geom_line(data = curve_ucar_theta, aes(x = sigma_3, y = sigma_1),
209     color = "orange", size = 1.2, linetype = "solid", alpha=0.7)
210   +
211   labs(
212     title = "Curva de rotura NGC+UC",
213     x = expression(sigma[3] ~ (MPa)),
214     y = expression(sigma[1] ~ (MPa))
215   ) +
216   xlim(-2,12)+ # edit limits to adjust plot IMPORTANT
217   ylim(0,45)+ # edit limits to adjust plot IMPORTANT
218   coord_fixed(ratio = 1/1) + # edit ratio to better visualize (sometimes
219   )
220   theme_gray()

```

Listing 1: Script to obtain the curves and tables for NGC and UC from known values of C_0 and T_0 .

C.2 Thermal fracturing simulation

Simulation Framework

The simulation models the thermo-mechanical response of a rock mass located at a depth of $z = 5000$ m subjected to progressive cooling, in order to determine the combinations of thermal expansion coefficient (θ) and temperature decrease (ΔT) that induce fracturing for a given value of Young's module.

Fixed input parameters (known values) include:

- Depth $z = 5000$ m.
- Effective unit weight of the overburden $\gamma_{\text{eff}} = 15 \text{ kN/m}^3$.
- Poisson's ratio $\nu = 0.25$.
- Initial temperature $T_{\text{ini}} = 450 \text{ K}$.
- Compressive ($C_0 = 25 \text{ MPa}$) and tensile ($T_0 = -2.5 \text{ MPa}$) strengths, along with the corresponding ratio $\xi = T_0/C_0$.

From these constants, additional model parameters are derived: the in-situ vertical stress $\sigma_{v,\text{ini}}$, the at-rest coefficient $k_0 = \nu/(1 - \nu)$, and the coefficients of the Úcar criterion (k_1, k_2) and New Griffith Criterion (NGC) (a, λ) based on empirical correlations with ξ .

Variables with defined ranges are:

- Thermal expansion coefficient $\theta \in [0.8 \times 10^{-6}, 1.5 \times 10^{-6}] \text{ K}^{-1}$.
- Young's modulus $E \in \{30, 40, 50, 60 \text{ GPa}\}$.
- Temperature decrease $\Delta T \in [0, 30] \text{ K}$.

Stress computation

For each $(\theta, E, \Delta T)$ combination, the cooling-induced thermal stress change is computed as: $\Delta\sigma_t = \frac{\theta E}{1-\nu} \Delta T$. This increment is subtracted from the initial vertical stress to obtain the updated principal stresses: $\sigma_v = \sigma_{v,\text{ini}} - \Delta\sigma_t$, $\sigma_h = k_0 \sigma_v - \Delta\sigma_t$. The major and minor principal effective stresses are then assigned as: $\sigma_1 = \max(\sigma_v, \sigma_h)$, $\sigma_3 = \min(\sigma_v, \sigma_h)$.

Fracture conditions

Failure is assessed using a combined criterion:

1. Compressive regime ($\sigma_3 > 0$): The Úcar criterion is applied, with fracture occurring when: $\sigma_1 \geq \sigma_{1,\text{UC}}(\sigma_3)$.
2. Tensile regime ($\sigma_3 \leq 0$):
 - If $\sigma_1 \geq \lambda|T_0|$, fracture is checked against the NGC expression: $\sigma_1 \geq a(\sigma_3 - T_0)^{3/4} + \lambda|T_0|$.
 - If $\sigma_1 < \lambda|T_0|$, fracture occurs when $\sigma_3 \leq T_0$.

Simulation output

For each E , the model identifies the minimum ΔT required to induce fracture for each θ . These critical $(\theta, \Delta T)$ pairs define fracture boundary curves in the ΔT – θ space. The output includes:

- A full dataset of stress states and fracture classification (“fails”/“no fail”).
- CSV export of all evaluated combinations.
- Graphical representation of the fracture boundaries for each E , enabling direct comparison of thermo-mechanical sensitivity across stiffness values.

```
1 # -----
2 # Thermo-mechanical simulation at 5000 m depth and
3 # evaluation against the NGC + UC criteria
4 # -----
5
6 # ---- Packages ----
7 suppressPackageStartupMessages({
8   library(dplyr)
9   library(tidyr)
10  library(purrr)
11  library(ggplot2)
12  library(readr)
13 })
14
15 # ---- Physical parameters ----
16 z <- 5000 # m
17 gamma_eff <- 15 # kN/m^3
18 nu <- 0.25
19 alpha_min <- 8e-6 # 1/K
```

```

20 alpha_max      <- 15e-6          # 1/K
21 E_vals         <- c(30e3, 40e3, 50e3, 60e3) # MPa (E1, E2)
22 T_ini          <- 450            # K (no explicitly in use)
23 k0             <- nu / (1 - nu)
24 C_0            <- 25             # MPa (sigma_c_adj)
25 T_0            <- -2.5           # MPa (sigma_t_adj)
26
27 # If no alternative fitted values are provided, use the ones
   abovesigma_c_adj <- C_0
28 sigma_t_adj     <- T_0
29 xi_adj         <- sigma_t_adj / sigma_c_adj # < 0 (traccia negativa)
30
31 # ---- NGC parameters derived from proposed correlations ----
32 # a = 0.531 * (1/|xi|)^0.75
33 # lambda = 0.472 * (1/|xi|)
34 a_param        <- 0.531 * (1 / abs(xi_adj))^0.75
35 lambda_param   <- 0.472 * (1 / abs(xi_adj))
36
37 # ---- Utilities: failure criteria ----
38 # UC (compressive): for sigma_3 > 0
39 # Normalizations and UC formula (in normalized form) using (k1, k2)
40 k1 <- (-(1 + abs(xi_adj)) + sqrt((1 + 7 * abs(xi_adj)) * (1 - abs(xi_adj)
   ))) / (2 * abs(xi_adj))
41 k2 <- (1 - k1 * (-xi_adj)) / sqrt(max(1e-4, -xi_adj))
42
43 sigma1_UC_from_sigma3 <- function(sigma3) {
44   sigma3_bar <- sigma3 / sigma_c_adj
45   sigma1_bar <- k1 * (sigma3_bar - xi_adj) + k2 * sqrt(pmax(0, sigma3_
   bar - xi_adj))
46   sigma1_bar * sigma_c_adj
47 }
48
49 # NGC (tensile / region with sigma_3 <= 0):
50 # sigma1_NGC = a*(sigma3 - T0)^(3/4) + lambda*|T0|, valid when sigma1
   >= lambda*|T0|
51 # If, in addition, sigma1_NGC < lambda*|T0|, compare sigma3 with T0:
   if sigma3 <= T0 -> failure
52 sigma1_NGC_from_sigma3 <- function(sigma3) {
53   # Note: beware of powers of negative numbers (sigma3 - T0) >= 0
   only when sigma3 >= T0
54   base <- pmax(0, sigma3 - T_0)
55   a_param * (base)^(3/4) + lambda_param * abs(T_0)
56 }
57
58 # Combined failure check:
59 fails_combined <- function(sigma1, sigma3) {
60   if (sigma3 > 0) {
61     # UC: failure occurs if sigma1 >= sigma1_UC
62     return(sigma1 >= sigma1_UC_from_sigma3(sigma3))
63   } else {
64     # NGC:
65     s1_NGC <- sigma1_NGC_from_sigma3(sigma3)
66     if (sigma1 >= lambda_param * abs(T_0)) {
67       # In the NGC branch: failure occurs if sigma1 >= sigma1_NGC
68       return(sigma1 >= s1_NGC)
69     } else {
70       # Below the threshold: pure tensile criterion
71       # failure occurs if sigma3 <= T0

```

```

72   return(sigma3 <= T_0)
73 }
74 }
75 }
76
77 # ---- In-situ and thermoelastic stress state ----
78 sigma_v_ini <- gamma_eff * z / 1000 # MPa (15 kN/m3 * 5000 m / 1000 =
79   75 MPa)
80
81 # Thermal variation
82 deltaT_grid <- seq(0, 30, by = 0.05) # K
83 alpha_grid <- seq(alpha_min, alpha_max, length.out = 100)
84
85 # Function to compute stresses for given (alpha, E, deltaT)
86 compute_stress <- function(alpha, E, deltaT) {
87   delta_sigma_t <- (alpha * E / (1 - nu)) * deltaT # MPa
88   sigma_v <- sigma_v_ini - delta_sigma_t
89   sigma_h <- k0 * sigma_v - delta_sigma_t
90   sigma1 <- max(sigma_v, sigma_h)
91   sigma3 <- min(sigma_v, sigma_h)
92   tibble(sigma_v = sigma_v, sigma_h = sigma_h, sigma1 = sigma1, sigma3 =
93     sigma3)
94 }
95
96 # ---- Full sweep for results and critical deltaT ----
97 # 1) Complete mesh table (for diagnostics/export)
98 # Recompute full mesh
99 resultados <- expand_grid(
100   E = E_vals,
101   alpha = alpha_grid,
102   deltaT = deltaT_grid
103 ) %>%
104   mutate(tmp = pmap(list(alpha, E, deltaT), compute_stress)) %>%
105   unnest(tmp) %>%
106   mutate(
107     regimen = ifelse(sigma3 > 0, "UC", "NGC"),
108     rompe = map_lgl(seq_len(n()), ~ fails_combined(sigma1[.x], sigma3
109       [.x]))
110   )
111
112 # Boundary (first deltaT causing failure for each (E, alpha))
113 criticos <- resultados %>%
114   group_by(E, alpha) %>%
115   arrange(deltaT, .by_group = TRUE) %>%
116   summarise(
117     deltaT_crit = if (any(rompe)) min(deltaT[rompe]) else NA_real_,
118     .groups = "drop"
119   )
120
121 # ---- Export to CSV ----
122 write_csv(resultados, "results_thermal_sim.csv")
123
124 # ---- Figure: failure curves (DeltaT* vs alpha) for each E ----
125 # Display only points with a valid solution (non-NA)
126
127 fig <- ggplot(
128   criticos %>% filter(!is.na(deltaT_crit)),
129   aes(x = alpha, y = deltaT_crit,

```



```

127     color = factor(E, levels = E_vals, labels = E_vals/1000),
128     group = factor(E))
129 ) +
130 geom_line(linewidth = 1) +
131 labs(
132   x = expression(alpha~"(K"^-1)*"),
133   y = expression(Delta*T[crit]~"(K)"),
134   color = "E (GPa)",
135   title = expression("Failure boundary (*Delta*T~" vs "*alpha*)" for
136     each "~E"),
137   subtitle = paste0(
138     "z = ", z, " m; k0 = ", round(k0, 3),
139     "; C0 = ", C_0, " MPa; T0 = ", T_0, " MPa; xi = ", signif(xi_
140       adj, 4)
141   )
142 ) +
143 theme_gray(base_size = 12)
144
145 print(fig)
146
147 # ---- Additional useful information in console ----
148 cat("Par metros NGC derivados: a =", signif(a_param, 5),
149     ", lambda =", signif(lambda_param, 5), "\n")
150 cat("CSV exported: results_thermal_sim.csv\n")

```

Listing 2: Script used to simulate thermo-mechanical response of a cooling rock.

Again, the R script is available on GitHub at [geotherm.examp.R](#).

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